MATHEMATICAL AND PHYSICAL MODELLING OF COMMUNICATIONS BETWEEN GPS/GNSS SATELLITES WITH ACCOUNT OF THE GENERAL RELATIVITY EFFECTS

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Abstract: *The intersatellite communications represent a new, important element in the contemporary conception about navigation. The main peculiarity of the propagation of a signal in a gravitational field is that the trajectory of the signal is curved. The report is a short review of the author's publications during the last 8 years on the problem about the exchange of signals between moving satellites, requiring the application of the methods of General Relativity Theory, as well as of the mathematical methods of algebraic geometry and elliptic integrals. New notions about "space-time distance" (which can be negative, equal to zero or positive), as well as about "geodesic distance" are introduced. Another new result is that the time for the propagation of the signal is expressed by means of elliptic integrals. The numerical calculation shows that the signal is propagating at a distance 26, 558.15102 [km] for a time 0.028134 [s]. This distance is comparable by order to the characteristic distance for intersatellite communications 49,465 [km].*

МАТЕМАТИЧЕСКО И ФИЗИЧЕСКО МОДЕЛИРАНЕ НА КОМУНИКАЦИИ МЕЖДУ GPS/GNSS СПЪТНИЦИ С ОТЧИТАНЕ ЕФЕКТИТЕ НА ОБЩАТА ТЕОРИЯ НА ОТНОСИТЕЛНОСТ

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Ключови думи: *междуспътникови комуникации; Приложна Обща Теория на Относителност; елиптични функции и интеграли; алгебрична геометрия*

Резюме: *Междуспътниковите комуникации са нов, съществен елемент на съвременната концепция за навигация. Главна особеност на разпространение на сигнали в гравитационно поле е, че траекторията им се изкривява . Докладът представлява кратък обзор на работите на автора през последните 8 години върху проблема за обмен на сигнали между движещи се спътници, изискващ прилагане методите на ОТО, както и на математически методи на алгебричната геометрия и елиптичните интеграли. Въведени са новите понятия за „пространствено-временно разстояние“ (може да бъде отрицателно, нулево или положително), също и „геодезично разстояние“. Друг нов резултат: изразяване на времето на разпространение на сигнала чрез елиптични интеграли. Численото пресмятане показва, че сигналът се разпространява на разстояние 26, 558.15102 [km] за време 0.028134 [s]. По порядък разстоянието е сравнимо с характерното разстояние за междуспътникови комуникации 49,465 [km].*

Introduction

Intersatellite communications are very important in view of creating a network of satellites and achieving high-rate data transfer [1], laser communications between satellites and relaying data around the globe with the purpose of early warning/tracking of missile launching [2].

Since 2003, after the review article [3] of Prof. Neil Ashby, the formalism of GRT has been commonly applied. A fundamental fact is that the trajectory of a signal (electromagnetic or laser), propagating in the near-Earth space is curved, due to which the signal is propagating at a greater time in comparison with the propagation time for the case of a flat space-time. Thus, this propagation time for the case of the curved space-time around the Earth is expressed by the simple formulae, well known in the literature as the Shapiro delay formulae

(1) $$T_{AB} = \frac{R_{AB}}{c} + \frac{2G_{\oplus}M_{\oplus}}{c^3}\ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right) .$$

In the above formulae the second term (the logarithmic correction) is related to the action of the gravitational field and is called the Shapiro time delay term. The propagation time T_{AB} for the propagation of the signal between two points A and B is obtained after integrating the s. c. null cone equation

(2) $$ds^2 = 0 = g_{00}c^2dT^2 + 2g_{0j}cdTdx^j + g_{ij}dx^idx^j$$

for the metric element in the near-Earth space

(3) $$ds^2 = -(c^2 + 2V)(dT)^2 + (1 - \frac{2V}{c^2})((dx)^2 + (dy)^2 + (dz)^2) .$$

Formulae (1) has several shortcomings, not making it possible to be applied for communications between satellites.

1. It depends on the initial and final moments of emission and reception of the signal and on the final distance of propagation of the signal, which is not known initially. No signal-receiving satellite is assumed to be situated at the final point. In this report and in all the papers [4, 5], [6] the aim will be to find the signal propagation timeT, so that the reception of the signal by the second satellite has to take into account not only the curving of the trajectory of the signal due to GRT-effects, but also the uncorrelated motion of the two satellites (on one orbit or on two different orbits).
2. Formulae (1) does not give the propagation time for concrete circumstances – for example, the propagation time of a signal, emitted by a satellite, moving on a plane elliptic orbit (characterized by the semi-major axis a, the ellipticity e and the eccentric anomaly angle E) or on a space-oriented orbit, characterized by 6 Kepler parameters with a dynamical parameter the angle of true anomaly f. For both cases, a qualitatively new result has been obtained [7], [8], [9] - the propagation time was expressed in terms of a combination of elliptic integrals of the first, second and the third kind (the first case of a plane orbit) or in terms of a combination of elliptic integrals of second and fourth order (for satellites, moving on space-oriented orbits).

In (1) the first term is the Euclidean distance, divided by the velocity of light **c.** However, in the framework of GRT the distance is given by the metric, so one cannot separate a "flat"-spacetime part and a "curved"-part. In [10] a modification of the Shapiro formulae (1) has been proposed

(4) $$T \approx \frac{1}{c}\int_{path}\sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}ds + \frac{2G_{\oplus}M_{\oplus}}{c^3}\int\frac{\sqrt{(\dot{x})^2+(\dot{y})^2+(\dot{z})^2}}{R_{AB}}dR_{AB} ,$$

where in the first term the distance is measured along a curve with a parameter *s* and $G_{\oplus}M_{\oplus}$ is the geocentric gravitational constant.

The algebraic geometry method of two null four-dimensional intersecting cones (case of plane elliptic orbits) – basic equations, newly introduced physical notions and some consequences

This method for calculation of the propagation time of a signal, emitted by a moving satellite and intercepted by another moving satellite has been proposed in the papers of the author [4, 5] and subsequently summarized in the review paper [6]. The basic idea is to write the null cone equation (2) for the metric (3) at two different space-time points

(5) $ds_1^2 = -(c^2 + 2V_1)(dT_1)^2 + \left(1 - \frac{2V_1}{c^2}\right)((dx_1)^2 + (dy_1)^2 + (dz_1)^2) = 0$,

(6) $ds_2^2 = -(c^2 + 2V_2)(dT_2)^2 + \left(1 - \frac{2V_2}{c^2}\right)((dx_2)^2 + (dy_2)^2 + (dz_2)^2) = 0$

and subsequently, to intersect these four-dimensional cones with the hyperplane, formed by the differential of the square of the Euclidean distance

(7) $dR_{AB}^2 = \mathrm{d}(x_1 - x_{2)})^2 + \mathrm{d}(y_1 - y_{2)})^2 + \mathrm{d}(z_1 - z_{2)})^2$.

For the case of plane elliptic orbits, expressions (5) – (7) represent a system of three (nonlinear) algebraic equations with respect to three variables - the differential of the square of the Euclidean distance and the differentials of the two propagation times. If the two satellites are on two-dimensional plane elliptic orbits, parametrized in the standard way as

(8) $x_1 = a_1(cosE_1 - e_1)$, $x_2 = a_2(cosE_2 - e_2)$, $y_1 = a_1\sqrt{1 - e_1^2}\sin E_1$, $y_2 = a_2\sqrt{1 - e_2^2}\sin E_2$,

then a differential equation in full derivatives is obtained with respect to the square of the Euclidean distance. The solution will no longer be equal to the Euclidean distance [4, 5], but will represent the s. c. "space-time" distance, which in analogy with the space-time interval in Special and General Relativity can be negative, equal to zero or positive. Let us take the limiting case

(9) $e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$.

Since the positions of the two satellites will coincide, the Euclidean distance will be zero, but the space-time distance (denoted below as $\hat{R}_{AB}^2$) will be non-zero and will be equal to

(10) $\hat{R}_{AB}^2 = 4a^2 sin^2 E.(1 - e^2) + a^2(e^2 - 2)$.

It can be positive, negative or equal to zero. The equality to zero is satisfied if

(11) $sin^2 E = \frac{2 - e^2}{4(1 - e^2)}$.

In fact, this condition in the papers [4, 5] is called "the compatibility condition for intersatellite communications", because it can be obtained after requiring the space-time distance to be comparable to the Euclidean distance. For the typical value of the eccentricity $\mathrm{e} = 0.01323881349526$ for the GPS orbit, the limiting value for the eccentric anomaly angle from the compatibility condition (10) is

(12) $E_{lim} = arcsin\left[\frac{1}{2}\sqrt{\frac{2 - e^2}{1 - e^2}}\right]$,

which has the numerical value 45.002510943228 [deg]. Although the eccentricity is taken for the GPS orbit, a disposition of satellites on one orbit (equal values for the large semi-major axis and equal eccentricities) for the above angular distance is typical for the Russian *GLONASS* satellite constellation (Global'naya Navigazionnaya Sputnikovaya Sistema) with 8 satellites, situated on one orbit. For the eccentricity of the GLONASS satellite orbits $e = 0.02$, the value for E slightly changes to $\mathrm{E} = 45.00573$ [deg]. Another restriction on the orbit eccentricity $\mathrm{e} \leq 0.81649658092$ follows from equality (11), since $sinE \leqq 1$ is fulfilled for $sinE$ as a trigonometric function. This means that for the Space-Ground Radio Interferometer Radio-Astron with a variable baseline, consisting of a satellite on an orbit with a large semi-major axis $a \approx 0.2 \times 10^6$ $[km]$ and a variable eccentricity of the satellite orbit, ranging from $e = 0.59$ to $e = 0.966$, the communications between satellites on the orbit are reliable only in the range $0.59 < \mathrm{e} \leq 0.81649658092$.

The third important physical notion, introduced in [4, 5] is about the "geodesic distance" – this is the real distance, travelled by the light or radio signal. Since this is a real distance, it should be positive and in accord with the physical essence of the Shapiro formulae (1), it should be greater also than the Euclidean distance. This is proved in a strict mathematical way for the general case in the papers [4, 5], because the geodesic distance is obtained after the "compatibility condition" is substituted in the formulae for the space-time distance. From the difference between the squares of the two distances the greatness of the geodesic distance (denoted with a $\sim$ (tilda) sign above) becomes evident

(13) R_{AB}^2 - $\tilde{R}_{AB}^2 = \frac{1}{2}(a_1^2 + a_2^2)$ - $e_1 e_2 a_1 a_2 + \frac{1}{4}(a_1^2 e_1^2 + a_2^2 e_2^2)$ - $2a_1 a_2\sqrt{(1-e_1^2)(1-e_2^2)}$.

In the limiting case of equal eccentricities and equal semi-major axis, this equality proves that the geodesic distance is greater than the Euclidean distance

(14) $\tilde{R}_{AB} = \sqrt{R_{AB}^2 + a^2(1-\frac{3}{2}e^2)}$.

Elliptic integrals for calculation of the propagation time of the signal for the case of satellites on plane elliptical orbits

Equations (5) and (6) represent differential equations with respect to the propagation time, defined for each of the satellites. The propagation time depends on variables, related to the motion of the satellites. For the case of the plane elliptic orbits, the dynamical parameters, related to the motion of the two satellites are the two eccentric anomaly angles E_1 and E_2. For the case of space-oriented orbits, the dynamical parameters will be the two true anomaly angles f_1 and f_2. All the other Kepler parameters, characterizing the orbit will be present in the formulas, but will not be dynamical ones. Moreover, each of the equations (5) and (6) will depend either on only E_1 or f_1, or on E_2 or f_2. The element of integration dr is lying in the plane of the orbit.

For a plane elliptic orbit, characterized by the parameters (a, e, E), the propagation time T is calculated in the paper [8] and then in the review article [7] of the author as

(15) $T = \frac{a}{c}\int\sqrt{1-e^2cos^2E}\,dE - \frac{2G_{\oplus}M_{\oplus}}{c^3}\int\sqrt{\frac{1+ecosE}{1-ecosE}}\,dE$.

It is important that the calculation turned out to be possible under the approximation

(16) $\beta = \frac{2V}{c^2} = \frac{2G_{\oplus}M_{\oplus}}{c^2 a} \ll 1$, $\frac{2G_{\oplus}M_{\oplus}}{c^2\, r_s} = 0.167.10^{-9}$, $G_{\oplus}M_{\oplus} = 3986005 \times 10^8 \left[\frac{m^3}{s^2}\right]$.

Physically related to weak gravitational fields and slow motion – these assumptions in gravitational physics are compatible with the Equivalence Principle [11] fields. Inequality (16) is obtained for the value of the velocity of light $c = 299{,}792.458\,[\frac{km}{s}]$ and value of the large semi-major axis $\mathrm{a} = 26{,}561\ [\mathrm{km}]$. It can be checked also that the coefficient in front of the first integral in (15) has the dimension of $[\frac{m}{\frac{m}{s}}]$ =[s], and the value of the coefficient in front of the second integral (the geocentric gravitational constant, divided by the third power of c) has the dimension $[\frac{\frac{m^3}{s^2}}{\frac{m^3}{s^3}}]$=[s]**.** Thus, the proper dimension for the coefficients in expression (15) for the propagation time T is a confirmation of the correctness of the mathematical formalism. If the inequality (16) is not taken into account, in [7] and [8] it was shown that yet a solution for T can be found, but in terms of an integral, which is not an abelian one and not possible to be solved analytically.

In (15) the first term is an elliptic integral of the second kind and the second term can be decomposed as a sum of an elliptic integral of the first kind and in the Weierstrass form (third-order polynomial under the square root in the denominator) [7, 8]

(17) $I_2^{(A)} = \frac{4G_{\oplus}M_{\oplus}}{c^3}.\frac{1}{\tilde{k}\sqrt{1-e^2}}.\int\frac{d\breve{y}}{\sqrt{\breve{y}(\breve{y}+1)(\breve{y}+\frac{1}{k^4})}}$

and an elliptic integral of the third kind, again in the Weierstrass form

(18) $I_2^{(B)} = \frac{4G_{\oplus}M_{\oplus}}{c^3 q^2}.\frac{1}{\sqrt{1-e^2}}\int\frac{d\breve{y}}{(\breve{y}_1-\frac{1}{q^2})\sqrt{\breve{y}_1(\breve{y}_1+1)(\breve{y}+\frac{1}{q^2})}}$.

Both integrals (17) and (18) are written in terms of variables, depending on $cosE$ and with coefficients, inversely proportional to the third power of the velocity of light c. Thus, their contribution will be much smaller than the contribution of the first integral in (15), which is inversely proportional to **c**. The numerical calculation (by means of online programs for numerical calculation of elliptic integrals) of the first integral in (15), performed by using the sixth iteration for eccentric anomaly angle

E_6 from the Kepler equation (see the details in the review article [7]) gives the numerical value for the leading term in (15) for the propagation time

(19) $$T_1^{(E_6)}=\frac{a}{c}\int_0^{E(6)}\sqrt{1-e^2cos^2E}\,dE=-0.0281342485273829\,[s]\ .$$

From the Kepler equation and for the sixth iteration E_6, the celestial time of motion for the satellite is calculated to be $T_{cel}=37.5082561\,[s]$. If the velocity of the satellite is taken to be $\mathrm{v}=3.874\,[\frac{km}{s}]$, the satellite will move at a distance $145.125\,[\mathrm{km}]$. If the propagation time for the signal (19) is multiplied by the light velocity (thus, the curving of the trajectory of the signal due to the gravitational field in the framework of GRT is taken into account), the signal will propagate at a distance $26{,}558.151016917626350\,[\mathrm{km}]$. Consequently, the propagation time is much smaller than the celestial time of motion of the satellite. This is so because the celestial time of motion is related to celestial mechanics, while the propagation time is an effect, following from General Relativity Theory.

Elliptic integrals of higher order for calculation of the propagation time of the signal for the case of satellites on space-oriented elliptical orbits

The position of space-oriented orbits is specified by 6 Kepler parameters $(a,e,\mathrm{I},\omega,\Omega,f)$ [9], which determine not only the position of the satellite on the orbit (related to the true anomaly angle f) and the parameters of the elliptic orbit (a,e), but also the position of the orbit in space $(\mathrm{I},\omega,\Omega)$, where I is the inclination of the orbit with respect to the equatorial plane. If at each point of the orbit the Cartesian coordinates are X,Y,Z, then the mapping $(X,Y,Z)\rightarrow(a,e,\mathrm{I},\omega,\Omega,f)$ from a topological point of view signifies a transition to a submersion manifold [10] (of 6 dimensions-more than the 3 dimensions of the initial manifold). The propagation time $\bar{T}$ for a signal, if emitted by a satellite on a space-oriented orbit with the only dynamical parameter-the true anomaly angle f is much more complicated (below i is the imaginary unit)

(20)
$$\bar{T}=i\left[-2\frac{na}{c}q^{\frac{3}{2}}+4\frac{G_{\oplus}M_{\oplus}n^2a}{c^4\,(1-e^2)^{\frac{3}{2}})}q^{\frac{3}{2}}\right]\tilde{J}_2^{(4)}(y,q)$$
$$+\,i\left[-\frac{2G_{\oplus}\ \ M_{\oplus}naq^{\frac{5}{2}}(1+e^2)}{c^3}\tilde{J}_2^{(4)}(\tilde{y},q)+\frac{2G_{\oplus}M_{\oplus}nq^{\frac{3}{2}}(1+e^2)}{c^3(1-e^2)}\tilde{J}_4^{(4)}(\tilde{y},q)\right],$$

where the second- and fourth- order elliptic integrals are expressed in the Legendre form

(21) $$\tilde{J}_2^{(4)}(y,q)=\int\frac{y^2dy}{\sqrt{(1-y^2)(1-q^2y^2)}}\ ,\quad \tilde{J}_4^{(4)}(\tilde{y},q)=\frac{q^5}{i}\int\frac{\hat{y}^4d\hat{y}}{\sqrt{(\hat{y}^2-1)(1-q^2\hat{y}^2)}}\ .$$

The variables $y,\tilde{y},\hat{y}$ and q are defined as

(22) $$y=\sqrt{\frac{(1+ecosE)}{q(1-ecosE)}}\ ,\quad \tilde{y}=\frac{\sqrt{1+2ecosf+e^2}}{1+e}\ ,\quad \hat{y}=\frac{\tilde{y}}{q}\ ,\quad q=\frac{1-e}{1+e}\ .$$

Conclusion

This paper has the purpose to review two major approaches, proposed in a series of papers in the last 8 years: the method of “four-dimensional intersecting null cones” and the elliptic integrals method for calculation of the propagation time of the signal. The previously introduced notions of “space-time distance”, “condition for intersatellite communication” and “geodesic distance” in fact refer to the case, when the two null cones are intersecting. Yet, the newly derived equality (14) for the geodesic distance and for the simplified case of equal eccentricities and semi-major axis clearly shows that that in this approach the property of the light signal to travel a greater distance is preserved, as it is the case for the Shapiro delay formulae (1). It is very interesting that the greater value for the geodesic distance in comparison with the Euclidean distance is related also with the restriction $\mathrm{e}\leq 0.81649658092$ on the eccentricity of the orbit and the limiting values $0.59<\ \mathrm{e}\leq\ 0.81649658092$ for the eccentricity of the orbit of the Space-Ground Radio Interferometer Radio-Astron. The last restriction is also natural for satellites on high elliptical orbits such as SBIRS (Space-Based Infrared System), consisting of four satellites, operating on geosynchronous Earth orbit, and sensors on two host satellites in a highly (large, with a large value of e) elliptical orbit. Elliptical orbit spacecrafts move with a variable orbit angular velocity, so it cannot be written as an explicit expression of a time function [12]. Yet, the given value for the eccentricity of highly elliptic orbit $\mathrm{e} = 0.7146$ in the monograph [12] falls well within the calculated in this paper range $\mathrm{e} \leq 0.81649658092$, so satellites can still operate and exchange signals.

The other important result in the papers [6], [8], [9] and in the review paper [7] concerns a new method for expressing the propagation time of the signal in terms of elliptic functions. The calculated propagation distance for the value of time $\mathrm{T} = 0.0281342485273829$ [s] in formulae (19) by using special online programs for numerical calculation of elliptic integrals is typical for the inter-satellite communications.

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